“Relationship between the Vehicle Routing Problem with Time Windows and the Assignment Problem”

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The paper considers a method of determining a solution to the Vehicle Routing Problem with Hard Time Windows which can be used in the Probabilistic Tabu Search Algorithm as the lower bound for the formation of the neighborhood of the potential solutions. If the objective function is independent of the vehicle idle before entering the next destination, the solution is an optimum of the original problem.

Introduction

The Vehicle Routing Problem with Hard Time Windows belongs to the class of NP-complete [1] for which used metaheuristic algorithms, in particular the various modifications of the Probabilistic Tabu Search algorithm[2]. These algorithms allow to find high-quality solutions, but they are not very useful in solving the Dynamic Routing Problems for which time of the search for the optimum is critical. If we analyze the oriented graph of the routing problem, in some cases, its structure is static and does not depend on time constraints, so the problem can be solved in polynomial time [3]. This solution can be used as the lower bound in the search of the optimum of the original problem. In the case when downtime of the vehicle before visiting the next customer does not affect the objective function, the solution is the optimum.

Problem statement

Let $C, dim(C) = n$ is a set of points corresponding to the current location of vehicles, $Q, dim(Q) = m$ is a set of destinations. Arrival at the destination should be done in the time window $[t_j, t_j + \Delta t_j]$. Time costs that may be associated with the unloading of goods equal to $w_j, \forall j \in Q$. $\Omega_{i,j}$ are the weight coefficients which determine the cost of moving from node i to node j.

We need to find the best routes of movement of vehicles to visit all the customers.

We take a sequence of matrices $\{X^k\}_{k=1}^n$, whose elements have the following values:

$$x_{i,j}^{(k)} = \begin{cases} 1, & \text{the } k\text{-vehicle is moving from } i \text{ to } j,\ i \in Q \cup \{k\}, j \in Q \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Let the waiting time of vehicle to the lower bound of the time window of the next destination will be determined by penalty function $\varphi(t)$. Then the objective function:

$$F(X) = \sum_{k \in C} [\sum_{i \in Q \cup \{k\}, j \in Q} (\Omega_{i,j} + \varphi(t_j - t_i - \omega_i - t_{i,j}) \cdot x_{i,j}^{(k)})] \to min \tag{2}$$

With constraints:

$$\sum_{k\in C}\sum_{j\in Q} x_{i,j}^{(k)} = 1, \forall i \in C \cup Q, \forall k \in C \tag{3}$$

$$\sum_{j\in Q}(x_{k,j}^{(k)} - x_{i,j}^{(k)}) \geq 0, \forall i \in Q, \forall k \in C \tag{4}$$

$$\sum_{i\in Q\cup\{k\}} x_{i,\omega}^{(k)} - \sum_{j\in Q} x_{\omega,j}^{(k)} = 0, \forall \omega \in Q, \forall k \in C \tag{5}$$

$$\sum_{i\in S}\sum_{j\notin S} x_{i,j}^{(k)} > 0, S = \{s \in Q : \sum_{j\in Q\cup C} x_{j,s}^{(k)} > 0\}, \forall k \in C \tag{6}$$

$$t_j \leq \tilde{t_j} \leq t_j + \Delta t_j, \forall j \in Q, \tilde{t_j}\text{- arrival time to the j-th destination} \tag{7}$$

Analysis of the graph structure

If the downtime of the vehicle before visiting the next customer does not affect the objective function ($\varphi = const = 0$), then time windows may not affect the structure of the graph and the problem reduces to the Assignment Problem in some cases.

Due to the fact that $NP \neq P$, single valued transition is impossible, but due to the time constraints graph can be acyclic and restrictions (6) and (7) can be dropped out after the removal of the arcs (i, j) for which is impossible to get to the node j regardless of the time of arrival to the i-th node.

In order to test the possibility of such transition, we introduce additional variables.

1. Considering restrictions on the objective function, the time of arrival to the i-th destination is equal:

$$\exists h_i \in [0, 1] : \tilde{t_i} = t_i + h_i \cdot \Delta t_i \tag{8}$$

2. Introduce the matrix U, the elements of which are shown in the form of the quantitative violation time window:

$$U(\vec{h}) : u_{i,j} = \begin{cases} 0, i = j, \\ t_i + h_i \cdot \Delta t_i + \omega_i + t_{i,j} - t_j - \Delta t_j, i \neq j \end{cases} \tag{9}$$

We can say that if the element of $U(\vec{h} = \vec{1})$ is negative, the movement between points is possible to perform not depending on time constraints. Therefore, we construct a graph G for the routing problem, which will already be taken into account constraints (6) and (7), and we can set the following incidence matrix:

$$I(\vec{h}) : I_{i,j} = \begin{cases} 0, & u_{i,j} \geq 0, \\ 1, & u_{i,j} < 0 \end{cases}, i \in C \cup Q, j \in Q \tag{10}$$

Defenition 1 *If:*

1. $\varphi = const = 0$;

2. the matrix $I(\vec{0})$ *is equal to the matrix* $I(\vec{1})$;

3. the graph G*, built on the the incidence matrix of* $I(\vec{0})$ *is not a multigraph, then the Vehicle Routing Problem with Hard Time Windows is reduced to the Assignment Problem.*

This is implied from the following lemmas.

Lemma 1 *If matrix* $I(\vec{0}) = I(\vec{1})$*, then* $I(\vec{0}) = I(\vec{h}), \forall \vec{h} : h_i \in [0,1]$

Proof. Elements of matrix $I_{i,j}$ do not depend on the vector $\vec{h}$ when $i \in C, j \in Q$, because the elements from the set C correspond to the location of cars at the initial time and they do not have time windows ($\Delta t_i = 0, \forall i \in C$). The remaining elements of the matrix can be represented as follows:

$$I_{i,j}(h_i) = \frac{1 - sign(u_{i,j}(h_i))}{2} \tag{11}$$

On the condition of the problem there is $I(\vec{0}) = I(\vec{1})$, therefore:

$$\frac{1 - sign(u_{i,j}(0))}{2} = \frac{1 - sign(u_{i,j}(1))}{2} \Rightarrow sign(u_{i,j}(0)) = sign(u_{i,j}(1))$$

Or:

$$\left[\begin{array}{l} \begin{cases} u_{i,j}(0) > 0 \\ u_{i,j}(1) > 0 \end{cases} \\ \begin{cases} u_{i,j}(0) < 0 \\ u_{i,j}(1) < 0 \end{cases} \end{array}\right.$$

The function $u_{i,j}(h_i)$ on the interval $h_i \in [0,1]$ is monotonically increasing, because the derivative $\frac{\partial u_{i,j}(h_i)}{\partial h_i} = \Delta t_i > 0$ is positive on the whole interval. Therefore:

$$\left[\begin{array}{l} \begin{cases} u_{i,j}(0) > 0 \\ u_{i,j}(1) > 0 \end{cases} \Rightarrow u_{i,j}(h_i) > 0 \\ \begin{cases} u_{i,j}(0) < 0 \\ u_{i,j}(1) < 0 \end{cases} \Rightarrow u_{i,j}(h_i) < 0 \end{array}\right. \Rightarrow sign(u_{i,j}(0)) = sign(u_{i,j}(h_i)), \forall h_i \in [0,1]$$

Then $sign(u_{i,j}(0)) = sign(u_{i,j}(h_i)) \Rightarrow I(\vec{h}) = I(\vec{0})$ ■

Lemma 2 *If the graph* G*, built on the incidence matrix* $I(\vec{h} = const)$ *is not a multigraph then this graph has no cycles.*

Proof by contradiction. Let there be given graph $G(V = C \cup Q, E)$, built on the incidence matrix $I(\vec{h} = const)$ and it has cycle:

$$E_{cycle} = \{(v_{s_1}, v_{s_2}), (v_{s_2}, v_{s_3}), ..., (v_{s_{k-1}}, v_{s_k}), (v_{s_k}, v_{s_1})\}$$

Let us assume that if the graph G is not multi-graph and it has not have arc (v_{s_1}, v_{s_k}), i.e. vehicle does not have time to get to vertex v_{s_k} from vertex v_{s_1} without violating the time window. Therefore:

$$t_{s_k} + \Delta t_{s_k} < \tilde{t}_{s_1} + \omega_{s_1} + t_{s_1,s_k} \tag{12}$$

By the assumption of the existence of the cycle E_{cycle} we have the following inequality($E_{route} = E_{cycle}/\{(v_{s_k}, v_{s_1})\}$):

$$t_{s_k} + \Delta t_{s_k} > \tilde{t}_{s_{k-1}} + \omega_{s_{k-1}} + t_{s_{k-1},s_k} = \tilde{t}_{s_1} + \sum_{(i,j)\in E_{route}} (t_{i,j} + \omega_{s_i}) = A \tag{13}$$

Let the movement from the vertex v_{s_1} to the vertex v_{s_k} will be implemented through the arcs E_{route}. Then:

$$\tilde{t}_{s_1} + \omega_{s_1} + t_{s_1,s_k} \le \tilde{t}_{s_1} + \omega_{s_1} + \sum_{(i,j)\in E_{route}} t_{i,j} \le A \tag{14}$$

Let us substitute (14) to (13):

$$t_{s_k} + \Delta t_{s_k} > \tilde{t}_{s_1} + \omega_{s_1} + t_{s_1,s_k} \tag{15}$$

A contradiction, consequently, the inequality (12) is incorrectly ■

The reduction to the assignment problem

If the rules from the definition 1 are satisfied, then it is possible reduce the initial problem to the assignment problem.

We introduce the change of variable X:

$$\forall k \in C: \; Y^k = \{I_{i,j}(0) \cdot x^{(k)}_{i,j}, i \in Q \cup \{k\}, j \in Q\} \tag{16}$$

Now the problem can be formulated in such way that it can be solved in polynomial time:

$$F(Y) = \sum_{k\in C} \sum_{i\in Q\cup\{k\}, j\in Q} \Omega_{i,j} \cdot y^{(k)}_{i,j}) \to min \tag{17}$$

$$\sum_{k\in C} \sum_{j\in Q} y^{(k)}_{i,j} = 1, \forall i \in C \cup Q, \forall k \in C \tag{18}$$

$$\sum_{j\in Q} (y^{(k)}_{k,j} - y^{(k)}_{i,j}) \ge 0, \forall i \in Q, \forall k \in C \tag{19}$$

$$\sum_{j\in Q\cup\{k\}} y^{(k)}_{i,\omega} - \sum_{i\in Q} y^{(k)}_{\omega,j} = 0, \forall \omega \in Q, \forall k \in C \tag{20}$$

The Assignment Problem in its classical form can be obtained by transforming the graph G at the bipartite graph $G_b(V_{begin}, V_{end}, E_b)$ on the following principle:

1. $V_{begin} = \{v \in C \cup Q \ : \exists (v, \omega) \in E, \omega \in Q\}$. It consists of a set of vertices such that there is at least one exiting arc from the set E;
2. $V_{end} = \{v \in Q \ : \exists (\omega, v) \in E, \omega \in C \cup Q\}$. It consists of a set of vertices such that there is at least one incoming arc of the set E;
3. $E_b = \{(v, \omega) \in E : v \in V_{begin}, \omega \in V_{end}\}$.

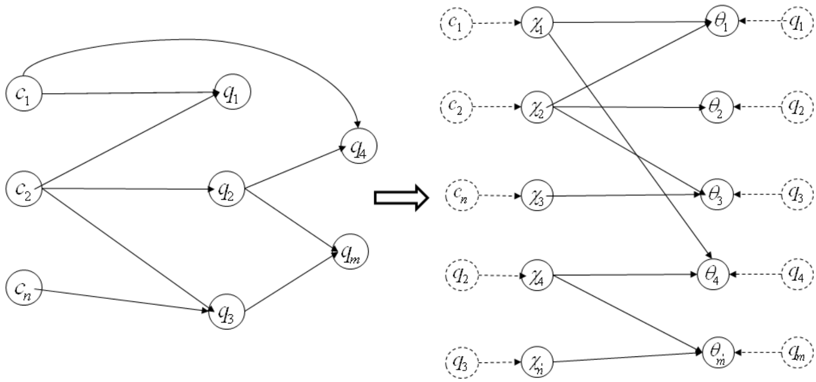

Figure 1. Example of converting the graph G into the bipartite graph G_b.

Results

The transformed problem (picture 1) recommended to be solved using the method of Goldberg and Kennedy, which based on the technique of scaling in reducing to the minimum-cost flow problem [4]. The complexity of the algorithm is $O(\sqrt{n} m log(nC))$.

If the function $\varphi(t)$ there is in the objective function, the solution of the transformed problem can be used how lower bound in a probabilistic tabu search algorithm in the formation of a neighborhood around the current solution.

Verification of the method was carried out on test problems Christofides, Goldberg and Teylarda [5, 6, 7] with time windows, satisfying the conditions that set out in the definition 1.

Conclusion

The paper proposed rules for converting the Vehicle Routing Problem with Hard Time Windows into the Assignment Problem. This approach should be used for the routing problems in which do not take into account the downtime of vehicles. Also the method can be used to initialize the metaheuristic algorithm, or as a lower bound for the determination of the optimum.

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